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Minimal Orthomodular Lattices from Quadratic Spaces over Finite Fields†

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If T is a finite, nonmodular, orthomodular lattice (OML), T is called minimal if all its proper subOMLs are modular. In this paper we give a new infinite list of minimal OMLs. They are obtained from quadratic spaces over a finite field K of cardinality $q \equiv 3 \pmod 4$. Their Greechie diagrams for $q = 7$ and $q = 11$ are presented in a new way.

1. THE LATTICES L AND T

We obtain orthomodular lattices (OMLs) by the following method. Let K be a finite field whose cardinality is $q = |K|$. Let us denote by $E = K^3$ the vector space over K; by L the modular lattice of all subspaces of E; by Q the canonical quadratic form over E; and by $\langle\, ,\rangle$ the corresponding inner product. If $u = (x, y, z)$ and $u' = (x', y', z')$, then $Q(u) = x^2 + y^2 + z^2$, $\langle u, u'\rangle = xx' + yy' + zz'$.

For $M \in L$, denote by $M^\perp$ the set of all $u \in E$ such that, for any $v \in M$, $\langle u, v\rangle = 0$.

A nonzero element u of E such that $Q(u) = 0$ is called an *isotropic vector.* A subspace M of E is said to be *isotropic* if the restriction of Q to E is singular. It is easy to see that a subspace M of E is isotropic if and only if there exists an isotropic vector ω such that $M = K\omega$ if M is one-dimensional, resp. $M = (K\omega)^\perp$ if M is two-dimensional.

We denote by T the set of all nonisotropic subspaces of E. Then $(T, \subset, \perp)$ is an OML, but T is not a sublattice of L.

†This paper is dedicated to the memory of Fred Rüttimann.

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We define the degree of an atom of T to be the number of blocks (maximal Boolean subalgebras of T) to which it belongs.

2. MINIMAL OMLs

We are interested in minimal OMLs. A finite OML is called minimal if it is nonmodular, and all its proper subOMLs are modular.

In Carréga (1998), infinitely many OMLs obtained from quadratic spaces over a field K of characteristic 2 are presented.

If $q = |K| = 2$, T is a minimal OML, in fact T is the horizontal sum of two Boolean algebras with two and three atoms, respectively.

For $q = 2^\alpha$, with α a prime number, T is not minimal, but it is the sum of a four-element Boolean algebra and a minimal OML T'. This OML T' has $q^2 - 1$ atoms and $q(q^2 - 1)/6$ blocks. All of its atoms are of the same degree, $q/2$.

The Greechie diagram of the OML T obtained for $q = 4$ is given in Fig. 1.

3. DESCRIPTION OF *T* IN ODD CHARACTERISTIC

In this section, we assume that $q = p^\alpha$ is a power of an odd prime number p. All OMLs T obtained in this case have similar properties. In all cases, T has q^2 atoms and $q(q^2 - 1)/6$ blocks. There are two kinds of atoms and two kinds of blocks. Some of the atoms, called weak, are of degree $(q - 1)/2$, and all other atoms, called stong, are of degree $(q + 1)/2$. There are $q(q + 1)/2$ weak atoms and $q(q - 1)/2$ strong atoms.

As regards blocks, there are two cases.

Case 1. If -1 is a square in K, which is equivalent to $q \equiv 1 \pmod 4$, each block of T is of one of the following forms:

where weak (resp., strong) atoms are white (resp., black).

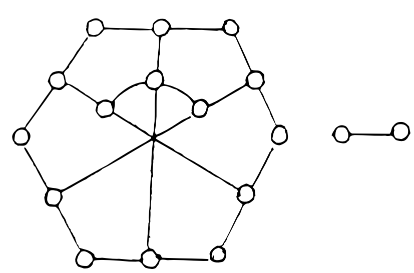

Fig. 1.

Case 2. If -1 is not a square in K, which is equivalent to $q \equiv 3$ (mod 4), each block is of one of the following forms:

In any case, each weak atom of T belongs to two different isotropic planes of E, whereas a strong atom does not belong to any isotropic plane. Actually, these planes of E are isotropic lines in the projective plane L. Thus, the weak atoms of T can be represented by the vertices of a kind of cobweb whose lines consist of the isotropic lines of L (see Fig. 2).

4. MINIMALITY

It is not difficult to see that for T to be minimal, it is necessary that q be prime. Our main result provides a new infinite list of minimal OMLs.

Theorem. If q is a prime number such that $q \geq 7$ and $q \equiv 3$ (mod 4), then T is a minimal OML.

In the proof of this result, we consider a nonmodular subOML T' of T and we prove that $T' = T$. For this, we need several technical lemmas, some of them using the polarity in projective plane L. It is necessary to build step by step the cobweb of T' (see Fig. 2). The largest part of the proof is needed

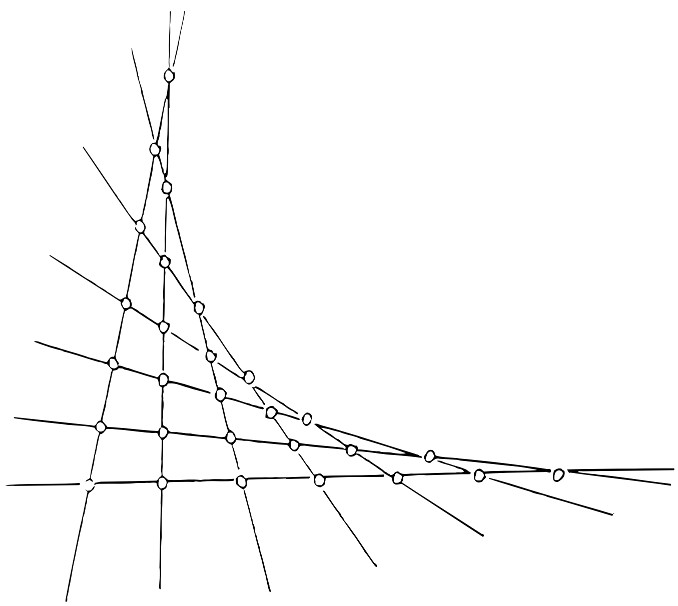

Fig. 2.

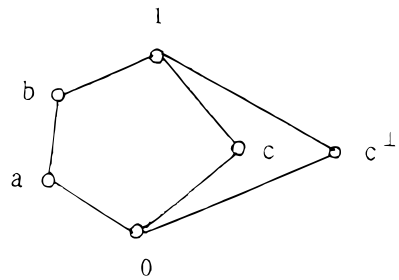

Fig. 3.

to show that there are at least seven lines in this cobweb, which allows us to prove in the last step that the number of these lines is q, and it follows that $T' = T$.

Corollary. The OML freely generated by the lattice given in Fig. 3 is infinite, even if we add the commutator conditions

$$\mathrm{Com}(a \wedge x, a \wedge y) = \mathrm{Com}(b^{\perp} \wedge x, b^{\perp} \wedge y) = 0$$

where $\mathrm{Com}(u, v) = (u \wedge v) \vee (u^{\perp} \wedge v) \vee (u \wedge v^{\perp}) \vee (u^{\perp} \wedge v^{\perp})$.

Remark. By computer we verified that, for $q = 5$, 13, and 17, T is minimal. We expect a general result for all prime numbers q such that $q \equiv$

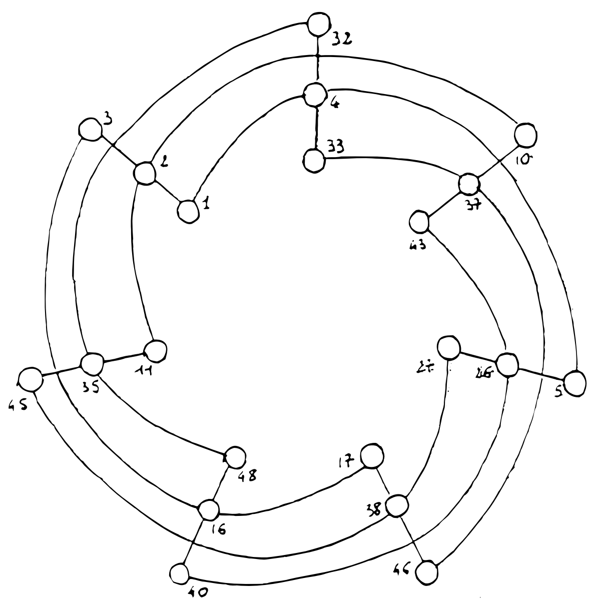

Fig. 4. $q = 7$, strong atom diagram.

1 (mod 4). But at this time we have only a few results about this case, which, at least in terms of the arguments discovered, seems quite different from the previous one.

Now we give the diagram of T in the two cases $q = 7$ (Figs. 4 and 5) and $q = 11$ (Figs. 6–8). These diagrams are given in a new way, separating the usual Greechie diagram into two diagrams corresponding to the strong and weak atoms, respectively. The diagram corresponding to the strong atoms is an ordinary Greechie diagram, and the other a graph with edges labeled by the atoms of the previous diagram; each edge corresponds to an incomplete block of T, and the label gives the missing atom of the block. In the case $q = 11$, the diagram of weak atoms, which is quite complicated, is itself separated into two diagrams (Figs. 7 and 8).

Table I gives some information on T in the two cases $q = 7$ and $q = 11$.

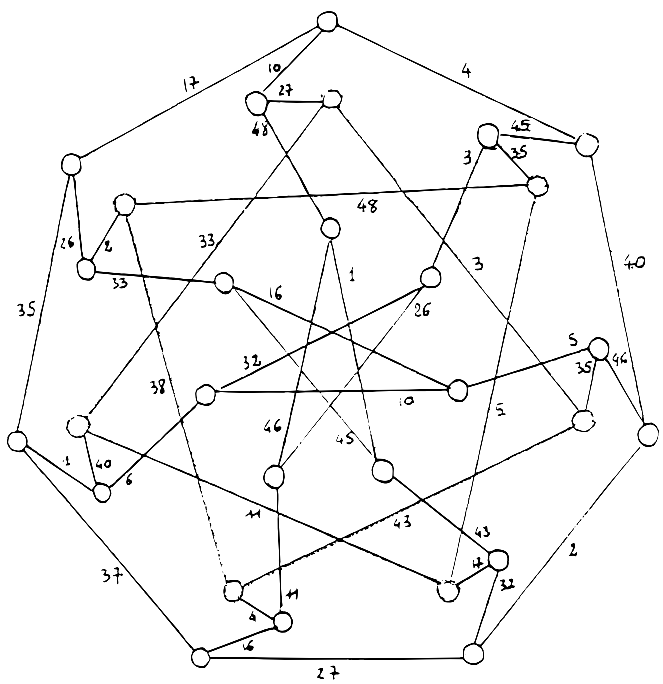

Fig. 5. $q = 7$, weak atom graph.

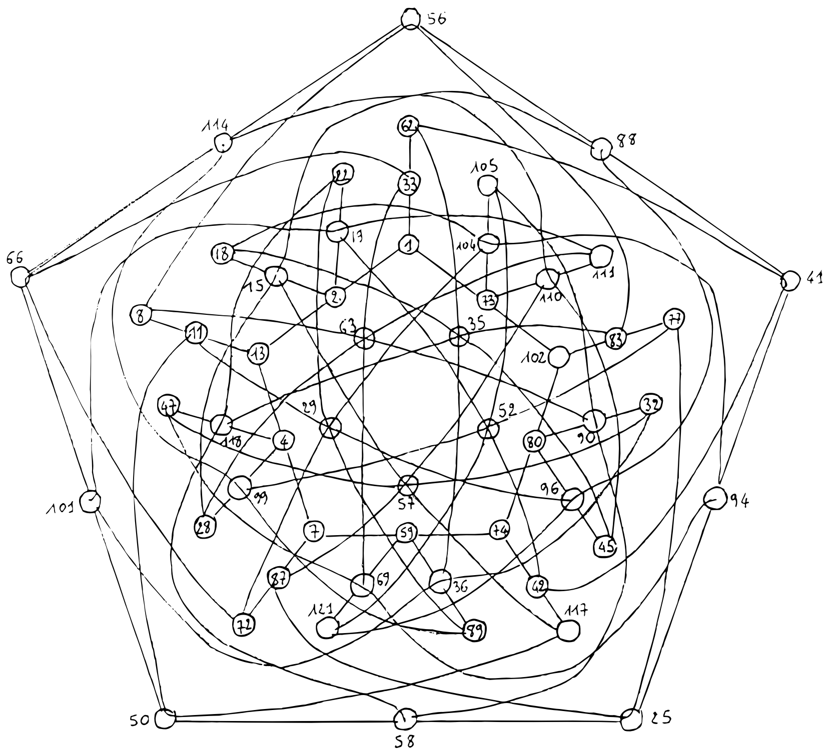

Fig. 6. $q = 11$, strong atom diagram.

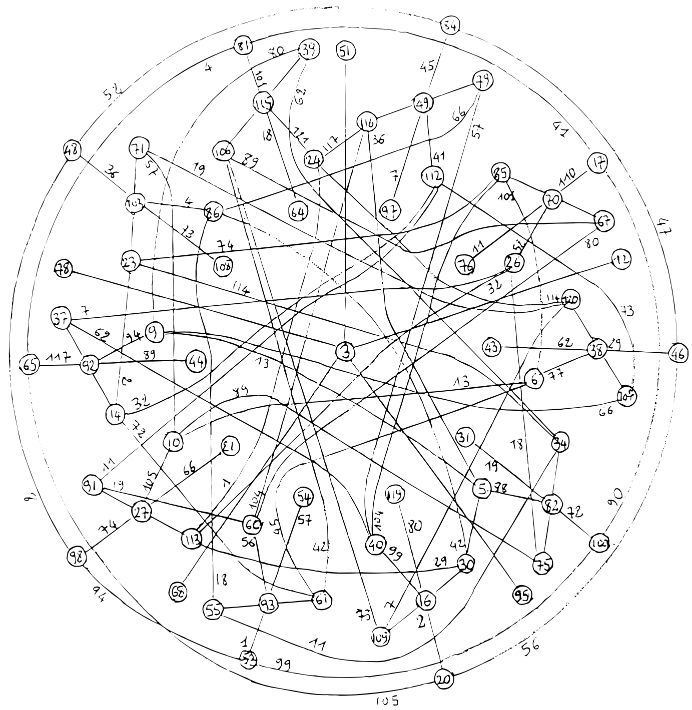

Fig. 7. $q = 11$, weak atom graph, first part of edges.

Table I

$\|K\|$	Atoms	Weak atoms	Strong atoms	Blocks
q	q^2	$q(q+1)/2$	$q(q-1)/2$	$q(q^2-1)/6$
7	49	28	21	56
11	121	66	55	220

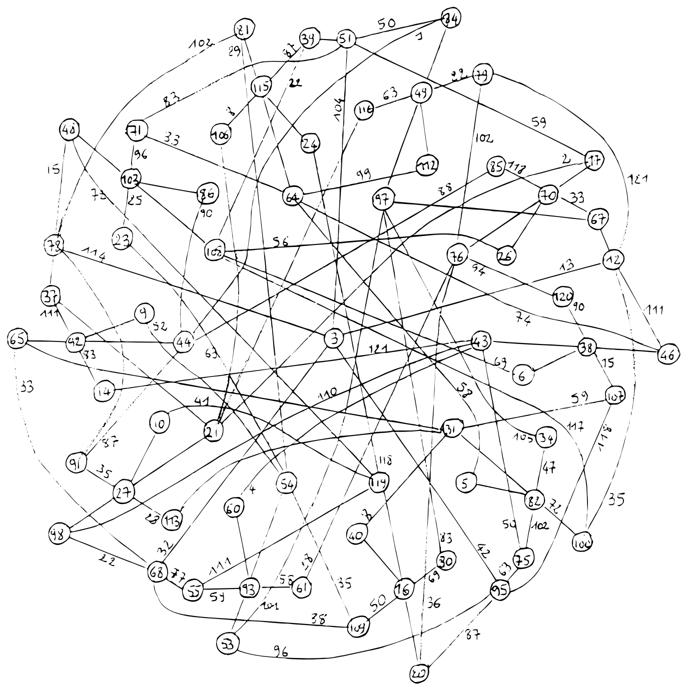

Fig. 8. $q = 11$, weak atom graph, second part of edges.

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